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Diagram cohomologies using categorical fibrations

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Abstract

We introduce a method for the construction of cohomology theories of diagrams of algebras by using pairings of categorical fibrations and show how it can be compared with the generalized Baues cohomology of a small category with coefficients in a natural system of complexes.

1. Introduction

This article presents a new approach to the construction of various cohomology theories of small categories. It is basically motivated by the following two observations:

(i) Gerstenhaber and Schack had described (in [4] and elsewhere) a cohomology theory for diagrams of algebras which is, in a sense, an extension of the Hochschild cohomology of associative algebras. It should be possible to perform analogous constructions for other cohomologies of algebraic structures, e.g. for the Hochschild-Mitchell cohomology (cf. [6]).

(ii) Other cohomologies of small categories (with different coefficient systems) seem to share the common pattern with the construction of Gerstenhaber and Schack, so it is probably possible to give a unified layout. In particular, it is desirable to include the very general construction of Baues and Wirsching of the cohomology with coefficients in a natural system (see [1]).

Finding connections between the construction of Gerstenhaber and Schack and that of Baues and Wirsching looks particulary appealing for it opens the possibility of pipelining the apparently disjoint calculation techniques from one to the other.

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At first sight both problems appear as relatively smooth ones, but they soon reveal difficulties in definition of morphisms (the first) and of their variance (the second). While investigating these, we found that there is much structure around and our present solution is quite far from what we expected in advance.

We begin the outline of the content just in the middle, describing the main construction. Its general setting are categorical fibrations. We introduce the notion of a pairing between a fibration and an opfibration with values in the category of cochain complexes (there is a partial analogy with the pairing between the tangent and the cotangent bundle in differential geometry). Then we use the fact that, given an arbitrary small category $\mathscr{I}$, a fibration $(\mathscr{E}, \mathsf{P}, \mathscr{B})$ induces a fibration $(\mathscr{E}^{\mathscr{I}}, \mathsf{P}^{\mathscr{I}}, \mathscr{B}^{\mathscr{I}})$ (and similarly for opfibrations) to extend the pairing to functor categories. This extension of pairings enables us to define a cohomology theory for diagrams, starting from a cohomology theory for objects. With this construction at hand, we show how to describe the cohomology for diagrams of algebras and then how to generalize it to the Hochschild–Mitchell cohomology of diagrams. Instead, one can try one's own favorite triple construction as well and combining the approach of [4] with the viewpoint of [3], the construction of deformation theories for diagrams of equationally defined classes of algebras becomes mere routine. Next we show how the cohomology with coefficients in a natural system fits in this frame, by constructing canonical fibration and opfibration over the category $\mathscr{I}$ and assigning a pairing which depends on the given natural system. The drawbacks of the second example are central to subsequent developments, because they indicate that our construction can be alternatively described as a cohomology of a small category with coefficients in a natural system of complexes. We immediately use this fact to extend some results of Baues and Wirsching (e.g. invariance of the construction with respect to the equivalence of categories). This is accomplished by applying a simple spectral sequence argument. In the last section we show how in an appropriate pairing situation the extension of pairings produces cohomology groups of spaces.

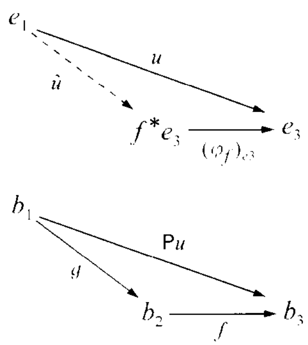
2. Preliminaries

2.1. Fibred categories

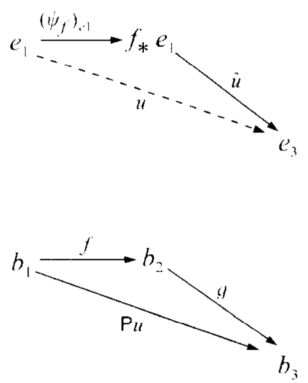
We recall some definitions and results from [5] and then give some examples for later use.

The *fiber* over $b \in \mathscr{B}$ of the functor $\mathsf{P} : \mathscr{E} \to \mathscr{B}$ is the subcategory $\mathscr{E}_b := \mathsf{P}^{-1}(b)$ consisting of all morphisms h in $\mathscr{E}$, such that $\mathsf{P}(h) = id_b$. Let $\mathsf{I}_b : \mathscr{E}_b \to \mathscr{E}$ be the inclusion functor. A *cleavage* (resp. *opcleavage*) for P consists of functors $f^* : \mathscr{E}_{b_2} \to \mathscr{E}_{b_1}$ ($f_* : \mathscr{E}_{b_1} \to \mathscr{E}_{b_2}$) for each morphism $f : b_1 \to b_2$ in $\mathscr{B}$, together with the natural transformations $\varphi_f : \mathsf{I}_{b_1} \circ f^* \to \mathsf{I}_{b_2}$ ($\psi_f : \mathsf{I}_{b_1} \to \mathsf{I}_{b_2} \circ f_*$) satisfying the following.

Axiom. $\mathsf{P}(\varphi_f) = f$ ($\mathsf{P}(\psi_f) = f$) and if $u : e_1 \to e_3$ satisfies $\mathsf{P}(u) = f \circ g$ ($\mathsf{P}(u) = g \circ f$) for some g, there is a unique $\widehat{u} : e_1 \to f^* e_3$ ($\widehat{u} : f_* e_1 \to e_3$) such that $\mathsf{P}(\widehat{u}) = g$ and $u = (\varphi_f)_{e_3} \circ \widehat{u}$ as in the diagram



(resp. for opcleavages, such that $\mathsf{P}(\widehat{u}) = g$ and $u = \widehat{u} \circ (\psi_f)_{e_1}$ as in the diagram



A *fibration* (*opfibration*) is a functor $\mathsf{P} : \mathscr{E} \to \mathscr{B}$ with cleavage (opcleavage). Now we consider examples of fibrations and opfibrations, some of which we shell need in the sequel.

Examples of fibrations. (i) The most familiar example of an (op)fibration is the projection of the category **MOD** of all modules to the category **Rng** of rings. Objects of **MOD** are pairs (R, M), where R is a ring and M is a (left) R-module, while the morphisms are pairs $(f, T) : (R, M) \to (S, N)$ where $f : R \to S$ is a ring morphism and $T : M \to N$ is a morphism of R-modules (we take on N the R-module structure induced by the pullback along f). The fiber over the ring R is simply a category of R-modules, while the cleavage over the ring morphism $f : R \to S$ is given by $f^*(S, N) := (R, f^* N)$, where $f^* N$ denotes the R-module given by the pullback along f, as before. For more details, see [5, p. 34].

(ii) We shall need a variant of the previous example. Fix a base ring R and denote by **Alg** the category of R-algebras and by **BIMOD** the category whose objects are pairs (A,M) with $A \in$ **Alg** and M an A-bimodule, and whose morphisms are pairs $(f,T) : (A,M) \to (B,N)$, similarly as before. The fibration is obviously given by the projection on the first factor and by the cleavage determined for every algebra homomorphism $f : A \to B$ by $f^*(B,N) := (A, f^*N)$, where f^*N is again the A-module obtained by pulling back the bimodule structure along f. This example will provide the basis for the cohomology of diagrams of algebras.

(iii) To describe the Hochschild–Mitchell cohomology of diagrams we need still another similar construction. Let **AddCat** denote the category of all small additive categories and **AddFun** the category of pairs $(\mathcal{A}, \mathsf{M})$, where $\mathcal{A}$ is a small additive category and M is a functor from the enveloping category $\mathcal{A}^e$ of $\mathcal{A}$ to the category of Abelian groups (cf. [6, p. 56]). A morphism $(\mathsf{F},\mu) : (\mathcal{A},\mathsf{M}) \to (\mathcal{A}',\mathsf{M}')$ consists of an additive functor $\mathsf{F} : \mathcal{A}^e \to \mathcal{A}'^e$ and a natural transformation $\mu : \mathsf{M} \to \mathsf{M}' \circ \mathsf{F}$. The fibration is the projection on the first factor and the cleavages are described as before (see also [6, pp. 153–154]).

(iv) In our final example we describe a quite different situation. Let $\mathscr{C}$ denote any category and let $\mathscr{C}^{\mathbf{2}}$ be the respective morphism category. The category $\mathscr{C}^{\mathbf{2}}$ has two natural projections on $\mathscr{C}$, the projection on the source (P_s) and on the target (P_t). We claim that $(\mathscr{C}^{\mathbf{2}}, \mathsf{P}_s, \mathscr{C})$ is a fibration and that $(\mathscr{C}^{\mathbf{2}}, \mathsf{P}_t, \mathscr{C})$ is an opfibration. Consider for example the second one: the fiber over an object $c \in \mathscr{C}$ is the category $\mathscr{C}/c$ ($\mathscr{C}$ over c) and for every $k : c \to c'$ in $\mathscr{C}$ we define the opcleavage by

$$k_*(f : a \to c) := k \circ f : a \to c' .$$

The construction of the cleavage in the first example is analogous.

Remark 1. We mention while passing that $(\mathscr{C}^{\mathbf{2}}, \mathsf{P}_s, \mathscr{C})$ is an opfibration iff $\mathscr{C}$ has pushouts and $(\mathscr{C}^{\mathbf{2}}, \mathsf{P}_t, \mathscr{C})$ is a fibration iff $\mathscr{C}$ has pullbacks (cf. [5, Example 2.12.]).

The main fact about fibrations and opfibrations is that they are closed under exponentiation: given a fibration $\mathsf{P} : \mathscr{E} \to \mathscr{B}$ and a small category $\mathscr{I}$ then $\mathsf{P}^{\mathscr{I}} : \mathscr{E}^{\mathscr{I}} \to \mathscr{B}^{\mathscr{I}}$ is also a fibration and similarly for opfibrations. For example, given functors $\mathsf{B}, \mathsf{B}' : \mathscr{I} \to \mathscr{B}$ and a natural transformation $\mu : \mathsf{B} \to \mathsf{B}'$ between them, then the cleavage $\mu^* : (\mathscr{E}^{\mathscr{I}})_{\mathsf{B}} \to (\mathscr{E}^{\mathscr{I}})_{\mathsf{B}'}$ is defined for $\mathsf{E} \in (\mathscr{E}^{\mathscr{I}})_{\mathsf{B}}$ by

$$(\mu^*\mathsf{E})i := (\mu_i)^*(\mathsf{E}i)$$

and

$$(\mu^*\mathsf{E})f := \widehat{\mathsf{E}f \circ \theta_{\mu_i}} \quad \text{(along } \theta_{\mu_i}\text{)},$$

where $\mathsf{E}\widehat{f \circ \theta_{\mu_i}}$ is the morphism obtained from the following diagram:

$$\begin{array}{ccc}
\mu_i^*\mathsf{E}i & \xrightarrow{\theta_{\mu_i}} & \mathsf{E}i \\
{\scriptstyle \mathsf{E}\widehat{f \circ \theta_{\mu_i}}}\downarrow & & \downarrow{\scriptstyle \mathsf{E}f} \\
(\mu_j)^*\mathsf{E}j & \xrightarrow{\theta_{\mu_j}} & \mathsf{E}j
\end{array}$$

$$\begin{array}{ccccc}
i & & \mathsf{B}'i & \xrightarrow{\mu_i} & \mathsf{B}i \\
{\scriptstyle f}\downarrow & & {\scriptstyle \mathsf{B}'f}\downarrow & & \downarrow{\scriptstyle \mathsf{B}f} \\
j & & \mathsf{B}'j & \xrightarrow{\mu_j} & \mathsf{B}j
\end{array}$$

as it can be deduced from the proofs of Theorems 2.10 and 3.6 in [5]. The description for opfibrations is analogous.

2.2. *Nerve of a category*

If $\mathcal{I}$ is a small category, then $\mathcal{I}$ determines a simplicial set called the *nerve* of the category and denoted $N(\mathcal{I})$. Its n-simplexes are the composable sequences of morphisms

$$\sigma = (a_0 \xleftarrow{f_1} a_1 \xleftarrow{f_2} a_2 \cdots a_{n-1} \xleftarrow{f_n} a_n)$$

of length n in $\mathcal{I}$. The face and the degeneracy maps are defined as follows:

$$\partial_i(f_1,\ldots,f_n) = \begin{cases} (f_2,\ldots,f_n) & i = 1, \\ (\ldots,f_i \circ f_{i+1},\ldots) & 1 < i < n, \\ (f_1,\ldots,f_{n-1}) & i = n \end{cases}$$

and

$$s_i(f_1,\ldots,f_n) = (\ldots,f_i, id, f_{i+1},\ldots).$$

To simplify the notation we will frequently use $\underline{\sigma} := \mathrm{dom}(f_n)$, $\overline{\sigma} := \mathrm{cod}(f_1)$, $|\sigma| := f_1 \circ \cdots \circ f_n$ and $(-1)^\sigma := (-1)^{\dim \sigma}$. Also, for a functor $\mathsf{F} : \mathcal{I} \to \mathcal{C}$ we will denote by $\mathsf{F}\sigma$ the chain

$$\mathsf{F}a_0 \xleftarrow{\mathsf{F}f_1} \mathsf{F}a_1 \xleftarrow{\mathsf{F}f_2} \cdots \xleftarrow{\mathsf{F}f_n} \mathsf{F}a_n$$

in $\mathcal{C}$.

When the small category $\mathscr{I}$ is ordered (i.e. when it has at most one morphism between any two objects) the nerve of $\mathscr{I}$ reduces to the usual simplicial complex associated to an ordered set. Its n-simplexes are chains $\sigma = (i_n \leq \cdots \leq i_0)$ of length $n+1$ and incidence is $\sigma \leq \sigma' \Leftrightarrow \sigma \subseteq \sigma'$.

3. Pairings of fibrations

Let **Cx** be the category of cochain complexes and cochain maps in **Ab**. A *pairing* between an opfibration $(\tilde{\mathscr{E}}, \mathsf{P}, \mathscr{B})$ and a fibration $(\mathscr{E}, \mathsf{P}, \mathscr{B})$ consists of contravariant–covariant bifunctors

$$\langle\ |\ \rangle_b : \tilde{\mathscr{E}}_b \times \mathscr{E}_b \to \mathbf{Cx}$$

for every $b \in \mathscr{B}$, such that for every $f : a \to b$ in $\mathscr{B}$, $\tilde{e}_a \in \tilde{\mathscr{E}}_a$ and $e_b \in \mathscr{E}_b$ there is a natural isomorphism

$$\langle \tilde{e}_a | f^* e_b \rangle_a \cong \langle f_* \tilde{e}_a | e_b \rangle_b$$

(i.e. the opcleavage and the cleavage are adjoint with respect to the pairing). The pth cochain group of this complex is denoted by $(\langle \tilde{e}_a | f^* e_b \rangle_a)^p$ and its coboundary by δ.

Our main objective in this section is to construct, starting from a pairing between $(\mathscr{E}, \mathsf{P}, \mathscr{B})$ and $(\tilde{\mathscr{E}}, \mathsf{P}, \mathscr{B})$ and a small category $\mathscr{I}$, a pairing between $(\mathscr{E}^{\mathscr{I}}, \mathsf{P}^{\mathscr{I}}, \mathscr{B}^{\mathscr{I}})$ and $(\tilde{\mathscr{E}}^{\mathscr{I}}, \mathsf{P}^{\mathscr{I}}, \mathscr{B}^{\mathscr{I}})$.

We begin by enlarging the meaning of the symbol $\langle\ |\ \rangle$. For $f : a \to b$ we define $\langle | \rangle_f : \tilde{\mathscr{E}}_a \times \mathscr{E}_b \to \mathbf{Cx}$ as

$$\langle \tilde{e}_a | e_b \rangle_f := \langle \tilde{e}_a | f^* e_b \rangle_a \cong \langle f_* \tilde{e}_a | e_b \rangle_b$$

and for a path $\sigma = (a_0 \xleftarrow{f_1} a_1 \cdots a_{n-1} \xleftarrow{f_n} a_n)$ we define $\langle | \rangle_\sigma : \tilde{\mathscr{E}}_{a_n} \times \mathscr{E}_{a_0} \to \mathbf{Cx}$ as

$$\langle \tilde{e}_{a_n} | e_{a_0} \rangle_\sigma := \langle \tilde{e}_{a_n} | e_{a_0} \rangle_{|\sigma|}.$$

Now, fix a small category $\mathscr{I}$ and choose a diagram $\mathsf{B} \in \mathscr{E}^{\mathscr{I}}$. Then take two diagrams $\tilde{\mathsf{E}} \in \tilde{\mathscr{E}}^{\mathscr{I}}$ and $\mathsf{E} \in \mathscr{E}^{\mathscr{I}}$, such that $\mathsf{B} = \mathsf{P}^{\mathscr{I}}(\tilde{\mathsf{E}}) = \mathsf{P}^{\mathscr{I}}(\mathsf{E})$, and define a bicomplex $(C^{p,q}, d_S, d_P)$

$$C^{p,q} := \left\{ \Gamma : N_q(\mathscr{I}) \to \coprod_{f:i \to j} (\langle \tilde{\mathsf{E}}i | \mathsf{E}j \rangle_{\mathsf{B}f})^p \mid \Gamma(\sigma) \in (\langle \tilde{\mathsf{E}}\underline{\sigma} | \mathsf{E}\overline{\sigma} \rangle_{\mathsf{B}\sigma})^p \right\}.$$

(The abstruse symbol $(\langle \tilde{\mathsf{E}}\underline{\sigma} | \mathsf{E}\overline{\sigma} \rangle_{\mathsf{B}\sigma})^p$ means: for a path σ in $\mathscr{I}$ starting at i and ending at j take the pth cochain group of the complex obtained by the pairing of $\tilde{\mathsf{E}}i$ with $\mathsf{E}j$ along the path $\mathsf{B}\sigma$.) The coboundary map $d_P : C^{p,q} \to C^{p+1,q}$ is induced by the pairing and can be easily described:

$$(d_{\mathrm{P}}\Gamma)(\sigma) := (-1)^{\sigma}\delta(\Gamma(\sigma)),$$

while the description of d_{S} is much more involved. The idea is to describe d_{S} as a simplicial boundary, approximatively of the form

$$(d_{\mathrm{S}}\Gamma)(\sigma) := \sum \pm\Gamma(\partial_i\sigma)\,.$$

As is usually the case there is a problem for $i = 1, q+1$ because $\Gamma(\partial_1\sigma)$ and $\Gamma(\partial_{q+1}\sigma)$ are not elements of $(\langle \tilde{\mathsf{E}}\underline{\sigma}|\mathsf{E}\overline{\sigma}\rangle_{\mathsf{B}\sigma})^p$ so we must map them in the right group using two homomorphisms which we call for convenience α_σ and β_σ.

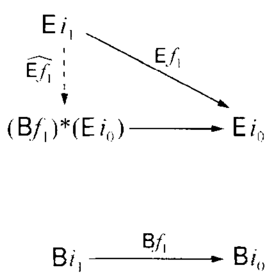
Given a q-simplex $\sigma = (i_0 \xleftarrow{f_1} i_1 \cdots i_{q-1} \xleftarrow{f_q} i_q) \in N_q(\mathscr{I})$ we first define the cochain map

$$\alpha_\sigma : \langle \tilde{\mathsf{E}}i_q|\mathsf{E}i_1\rangle_{\mathsf{B}(\partial_1\sigma)} \to \langle \tilde{\mathsf{E}}i_q|\mathsf{E}i_0\rangle_{\mathsf{B}\sigma}.$$

Applying the functor

$$\langle \tilde{\mathsf{E}}i_q|\ \ \rangle_{\mathsf{B}(\partial_1\sigma)} : \mathscr{E}_{\mathsf{B}i_1} \to \mathbf{Cx}$$

to the diagram



we obtain the compositum

$$\langle \tilde{\mathsf{E}}i_q|\mathsf{E}i_1\rangle_{\mathsf{B}(\partial_1\sigma)} \xrightarrow{\langle \tilde{\mathsf{E}}i_q|\widehat{\mathsf{E}f_1}\rangle_{\mathsf{B}(\partial_1\sigma)}} \langle \tilde{\mathsf{E}}i_q|(\mathsf{B}f_1)^*\mathsf{E}i_0\rangle_{\mathsf{B}(\partial_1\sigma)} \cong \langle \tilde{\mathsf{E}}i_q|\mathsf{E}i_0\rangle_{\mathsf{B}\sigma}$$

which we call α_σ.

Similarly, the application of the contravariant functor

$$\langle\ \ |\mathsf{E}i_0\rangle_{\mathsf{B}(\partial_q\sigma)} : \tilde{\mathscr{E}}_{\mathsf{B}i_{q-1}} \to \mathbf{Cx}$$

to the diagram

$$\begin{array}{ccc} \tilde{\mathsf{E}}i_q & \longrightarrow & (\mathsf{B}f_q)^*(\tilde{\mathsf{E}}i_q) \\ & \searrow^{\tilde{\mathsf{E}}f_q} & \Big\downarrow{\scriptstyle \widehat{\tilde{\mathsf{E}}f_q}} \\ & & \tilde{\mathsf{E}}i_{q-1} \end{array}$$

$$\mathsf{B}i_q \xrightarrow{\mathsf{B}f_q} \mathsf{B}i_{q-1}$$

yields the compositum

$$\langle \tilde{\mathsf{E}}i_{q-1}|\mathsf{E}i_0\rangle_{\mathsf{B}(\partial_q\sigma)} \xrightarrow{\widehat{\langle \tilde{\mathsf{E}}f_q|\mathsf{E}i_0\rangle_{\mathsf{B}(\partial_q\sigma)}}} \langle (\mathsf{B}f_q)_*\tilde{\mathsf{E}}i_q|\mathsf{E}i_0\rangle_{\mathsf{B}(\partial_q\sigma)} \cong \langle \tilde{\mathsf{E}}i_q|\mathsf{E}i_0\rangle_{\mathsf{B}\sigma}$$

which we denote by β_σ.

Using α and β we can describe the coboundary $d_S : C^{p,q} \to C^{p,q+1}$:

$$(d_S\Gamma)(\sigma) := \alpha_\sigma(\Gamma(\partial_1\sigma)) + \sum_{i=2}^{q}(-1)^i\Gamma(\partial_i\sigma) + (-1)^{q+1}\beta_\sigma(\Gamma(\partial_{q+1}\sigma))$$

Now, α and β are cochain maps so it is easy to verify that $d_Sd_S = 0$ and $d_Sd_P = d_Pd_S$, therefore we can finally define

$$(\langle \tilde{\mathsf{E}}|\mathsf{E}\rangle_{\mathsf{B}}, \delta) := \mathrm{Tot}(C^{p,q}, d_S, d_P)\,.$$

In the remaining part of this section we show that the extension of a pairing is again a pairing. We must prove the contravariant–covariant bifunctoriality and the 'adjointness' between the cleavages and the opcleavages. In both cases we will compare the defining bicomplexes and to fix the ideas we will always choose $\sigma \in N_q\mathscr{I}$ with $\underline{\sigma} = i$, $\overline{\sigma} = j$ and $|\sigma| = f$ as a test q-simplex. Now, take $\mathsf{B} \in \mathscr{B}^{\mathscr{I}}$, $\mathsf{E} \in \tilde{\mathscr{E}}^{\mathscr{I}}$ and a morphism $\mu : \mathsf{E} \to \mathsf{E}'$ in $(\mathscr{E}^{\mathscr{I}})_{\mathsf{B}}$ and define the map

$$F^{*,*} : C^{*,*}(\tilde{\mathsf{E}},\mathsf{E}) \to C^{*,*}(\tilde{\mathsf{E}},\mathsf{E}')$$

as follows:

$$(F^{p,q}\Gamma)\sigma := \langle(\mathsf{B}f)_*(\mathsf{E}i)|\mu_j\rangle^p_{\mathsf{B}i}(\Gamma\sigma).$$

The morphism $F^{*,*}$ is defined using a chain morphism, so it obviously commutes with d_P, while a straightforward calculation shows, that it commutes with d_S too, therefore it induces the morphism

$$\langle \tilde{\mathsf{E}}|\mu\rangle_{\mathsf{B}} : \langle \tilde{\mathsf{E}}|\mathsf{E}\rangle_{\mathsf{B}} \to \langle \tilde{\mathsf{E}}|\mathsf{E}'\rangle_{\mathsf{B}}.$$

An analogous construction establishes the contravariant functoriality in the second factor. Toward the proof of adjointness, consider a natural morphism $\nu : \mathsf{B} \to \mathsf{B}'$ in $\mathscr{B}^{\mathscr{I}}$ and functors $\tilde{\mathsf{E}} \in (\tilde{\mathscr{E}}^{\mathscr{I}})_{\mathsf{B}}$, $\mathsf{E} \in (\mathscr{E}^{\mathscr{I}})_{\mathsf{B}'}$. We claim that

$$\langle \tilde{\mathsf{E}} | \nu^* \mathsf{E} \rangle_{\mathsf{B}} \cong \langle \nu_* \tilde{\mathsf{E}} | \mathsf{E} \rangle_{\mathsf{B}'}.$$

As mentioned in Section 2.1, $\nu^*\mathsf{E}$ is determined by $(\nu^*\mathsf{E})i := (\nu_i)^*(\mathsf{E}i)$ and similarly for ν_* so we have (with $\sigma \in N_q\mathscr{I}$ as above):

$$\langle \tilde{\mathsf{E}}\underline{\sigma} | (\nu^*\mathsf{E})\overline{\sigma} \rangle_{\mathsf{B}\sigma} = \langle \tilde{\mathsf{E}}i | (\nu_j)^*(\mathsf{E}j) \rangle_{\mathsf{B}f} \cong \langle \tilde{\mathsf{E}}i | (\mathsf{B}f)^*(\nu_j)^*(\mathsf{E}j) \rangle_{\mathsf{B}i}$$

and

$$\langle (\nu_*\tilde{\mathsf{E}})\underline{\sigma} | \mathsf{E}\overline{\sigma} \rangle_{\mathsf{B}'\sigma} = \langle (\nu_i)_* \tilde{\mathsf{E}}i | (\mathsf{B}'f)^*(\mathsf{E}j) \rangle_{\mathsf{B}'i} \cong \langle \tilde{\mathsf{E}}i | (\nu_i)^*(\mathsf{B}'f)(\mathsf{E}j) \rangle_{\mathsf{B}i}.$$

The naturality of ν gives $(\mathsf{B}f)^*(\nu_j)^* = (\nu_i)^*(\mathsf{B}'f)^*$, therefore

$$\langle \tilde{\mathsf{E}}\underline{\sigma} | (\nu^*\mathsf{E})\overline{\sigma} \rangle_{\mathsf{B}\sigma} \cong \langle (\nu_*\tilde{\mathsf{E}})\underline{\sigma} | \mathsf{E}\overline{\sigma} \rangle_{\mathsf{B}'\sigma}$$

for every $\sigma \in N_*\mathscr{I}$ and that yields the isomorphism between $\langle \tilde{\mathsf{E}} | \nu^*\mathsf{E} \rangle_{\mathsf{B}}$ and $\langle \nu_*\tilde{\mathsf{E}} | \mathsf{E} \rangle_{\mathsf{B}'}$.

4. The Hochschild cohomology of diagrams

In this section we show how the cohomology of diagrams of algebras fits in the described construction.

We first recall the original definition given in [4]. Let $\mathscr{I}$ be a small ordered category, and let $\mathsf{A} : \mathscr{I} \to \mathbf{Alg}$ (algebras over a fixed ring R) and $\mathsf{M} : \mathscr{I} \to \mathbf{Ab}$ be functors such that for every $i \in \mathscr{I}$, $\mathsf{M}i$ is an $\mathsf{A}i$-bimodule and for every $i \le j$ in $\mathscr{I}$ the following relation holds:

$$\mathsf{M}_{ij}(amb) = \mathsf{A}_{ij}(a)\mathsf{M}_{ij}(m)\mathsf{A}_{ij}(b) \quad (a, b \in \mathsf{A}i,\ m \in \mathsf{M}i)\,.$$

In other words, we require that M_{ij} is an A_i-bimodule homomorphism, where $\mathsf{M}j$ is an $\mathsf{A}i$-bimodule through $\mathsf{A}_{ij} : \mathsf{A}i \to \mathsf{A}j$. If this holds, we say that M is a bimodule over the diagram of algebras A. Now we describe the cohomology of A with coefficients in the A-bimodule M (see [4, p. 15]). Let

$$C^{p,q}(\mathsf{A},\mathsf{M}) := \left\{ \Gamma : \Sigma_q(\mathscr{I}) \to \coprod_{i \le j} C^p(\mathsf{A}i, \mathsf{M}j) \mid \Gamma(\sigma) \in C^p(\mathsf{A}\underline{\sigma}, \mathsf{M}\overline{\sigma}) \right\},$$

where $C^p(\mathsf{A}\underline{\sigma}, \mathsf{M}\overline{\sigma})$ is the Hochschild cochain complex, and two commuting coboundaries $d_{\mathrm{H}} : C^{p,q} \to C^{p+1,q}$ and $d_{\mathrm{S}} : C^{p,q} \to C^{p,q+1}$, which are defined as follows:

$$(d_{\mathrm{H}}\Gamma)(\sigma) := (-1)^{\sigma}\delta\Gamma(\sigma) \quad (\delta \text{ is the Hochschild coboundary})$$

and for $\sigma = (i_{q+1} \le \cdots \le i_0) \in \Sigma_q(\mathscr{I})$

$$(d_{\mathrm{S}}\Gamma)(\sigma) := \mathsf{M}_{i_1 i_0} \circ \Gamma(\partial_0\sigma) + \sum_{k=1}^{q} (-1)^k \Gamma(\partial_k\sigma) + (-1)^{q+1}\Gamma(\partial_0\sigma) \circ \mathsf{A}_{i_{q+1} i_q}.$$

Then $C^*(\mathsf{A},\mathsf{M})$ is given by

$$C^*(\mathsf{A},\mathsf{M}) := \mathrm{Tot}(C^{*,*}(\mathsf{A},\mathsf{M}), d_{\mathrm{S}}, d_{\mathrm{H}}).$$

(The authors call the cohomology of $C^*(\mathsf{A},\mathsf{M})$ simply diagram cohomology, but we prefer to call it Hochschild cohomology of the diagram A with coefficients in M, to distinguish it from other diagram cohomologies.)

Next we show how to obtain the Hochschild cohomology of the pair (A,M) from the ordinary Hochschild cohomology, using extension of pairings. Remember that we have to describe a fibration, an opfibration and a pairing between them. We just take the fibration $(\mathbf{BIMOD},\mathsf{P},\mathbf{Alg})$ from the second example of Section 2.1 and the opfibration $(\mathbf{Alg^2},\mathsf{P}_1,\mathbf{Alg})$ from the fourth example of the same section. The description of the pairing is straightforward:

$$\langle (f : A' \to A) | (A,M) \rangle_A := C^*(A', f^*M),$$

where $C^*(A', f^*M)$ is the Hochschild cochain complex of the algebra A' with coefficients in M, taken as A'-bimodule through f. For a map $k : A \to B$ we have

$$\langle k_* f | (B,M) \rangle_B = \langle k \circ f | (B,M) \rangle_B = C^*(A', (k \circ f)^* M) = \langle f | k^*(B,M) \rangle_A .$$

The map $k : f \to g$ from $f : A' \to A$ to $g : A'' \to A$ in $\mathbf{Alg}/A$ ($g \circ k = f$), induces the map $\langle k | (A,M) \rangle_A : C^*(A'', g^*M) \to C^*(A', f^*M)$ which is simply the composition with k on the right, and similarly for the second factor, where we obtain the composition on the left. This proves the bifunctoriality of the pairing.

It remains to prove that the Hochschild cohomology of the diagram A with coefficients in the A-bimodule M is a particular case of the previously described construction. First, we define $\widehat{\mathsf{A}} : \mathscr{I} \to \mathbf{Alg^2}$ by $\widehat{\mathsf{A}}i := id_{\mathsf{A}i}$ and $\widehat{\mathsf{A}}(f) := (\mathsf{A}f, \mathsf{A}f)$. Similarly, we substitute M with a functor $\widehat{\mathsf{M}} : \mathscr{I} \to \mathbf{BIMOD}$ determined by $\widehat{\mathsf{M}}i := (\mathsf{A}i, \mathsf{M}i)$ and $\mathsf{M}f := (\mathsf{A}f, \mathsf{M}f)$. (Note that we have in this way described embeddings of diagrams of algebras into $(\mathbf{Alg^2})^{\mathscr{I}}$ and of A-bimodules into $\mathbf{BIMOD}^{\mathscr{I}}$, such that $\mathsf{P}_l(\widehat{\mathsf{A}}) = \mathsf{P}(\widehat{\mathsf{M}}) = \mathsf{A}$.) We claim that

$$(\langle \widehat{\mathsf{A}} | \widehat{\mathsf{M}} \rangle_{\mathsf{A}}, \delta) = \mathrm{Tot}(C^{p,q}(\mathsf{A},\mathsf{M}), d_{\mathrm{S}}, d_{\mathrm{H}}) = C^*(\mathsf{A},\mathsf{M}) .$$

According to the construction, $(\langle \widehat{\mathsf{A}} | \widehat{\mathsf{M}} \rangle_{\mathsf{A}}, \delta)$ is the total complex of the bicomplex $(C^{p,q}, d_{\mathrm{S}}, d_{\mathrm{P}})$, where

$$C^{p,q} = \left\{ \Gamma : \Sigma_q(\mathscr{I}) \to \prod_{i \le j} C^p(\mathsf{A}i, \mathsf{M}j) \mid \Gamma(i_q, \ldots, i_0) \in C^p(\mathsf{A}i_0, \mathsf{M}i_q) \right\} .$$

(Here we have already taken into account that $\mathscr{I}$ is an ordered category and that the pairing is defined using Hochschild cochain complex.) Moreover, evidently $d_{\mathrm{P}} = d_{\mathrm{H}}$, so it will suffice to show that the two definitions of d_{S} coincide. To this end it is necessary to calculate morphisms α and β. By the definition, the morphism β_σ is obtained by applying the functor $\langle id_{\mathsf{A}i_{q+1}} | \ \rangle_{\mathsf{A}_{i_q \cdots i_1}}$ to the map $(id_{\mathsf{A}i_1}, \mathsf{M}_{i_1 i_0}) : (\mathsf{A}i_1, \mathsf{M}i_1) \to$

$(\mathsf{A}i_1, \mathsf{A}^*_{i_1 i_0}\mathsf{M}i_0)$. The result is the map $\beta_\sigma : C^*(\mathsf{A}i_{q-1}, \mathsf{A}^*_{i_q i_1}\mathsf{M}i_1) \to C^*(\mathsf{A}i_{q+1}, \mathsf{A}^*_{i_q i_0}\mathsf{M}i_0)$ which is, by the definition of the pairing exactly the composition with $\mathsf{M}_{i_1 i_0}$ on the right. In the same way we find that the map α is given by the composition with $\mathsf{A}_{i_{q+1} i_q}$ on the left. This proves that the two definitions of d_S coincide.

Among other problems that motivated our work, we tried to extend the classical Hochschild–Mitchell cohomology of generalized rings ('rings with several objects', cf. [6]) to diagrams of such rings. Using the technique of pairings this became almost trivial.

As the fibration we take (**AddFun**, P, **AddCat**) (Example 3 in Section 2.1), and as the corresponding opfibration ($\mathbf{AdCat^2}$, P, **AdCat**). The pairing is constructed similarly as in our first example:

$$\langle(\mathsf{F} : \mathscr{I}' \to \mathscr{I})|(\mathscr{I}, \mathsf{M})\rangle_{\mathscr{I}} := C^*(\mathscr{I}', \mathsf{F}^*\mathsf{M}),$$

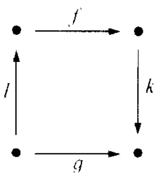
where $C^*(\mathscr{I}', \mathsf{F}^*\mathsf{M})$ is the usual Hochschild–Mitchell cochain complex (cf. [6, pp. 71–73]). The description of the Hochschild–Mitchell cohomology of a diagram A of (generalized) rings with coefficients in an A-bimodule M goes as before: substitute A and M by $\widehat{\mathsf{A}}$ and $\widehat{\mathsf{M}}$, respectively, and define

$$H^*(\mathsf{A}, \mathsf{M}) := H^*(\langle\widehat{\mathsf{A}}|\widehat{\mathsf{M}}\rangle).$$

5. Extensions of pairings and natural systems

We begin this section with the description of the original construction of Baues and Wirsching (cf. [1]) and than show that the extension of pairings can be viewed as its generalization altough in the way quite different from the one described in Section 4.

Let $\mathscr{I}$ be a small category. The *category of factorizations in* $\mathscr{I}$, denoted by $F\mathscr{I}$, is given as follows: its objects are morphisms of $\mathscr{I}$ and morphisms $f \to g$ are all pairs (k, l) of morphisms in $\mathscr{I}$ for which the diagram



commutes in $\mathscr{I}$. Hence $g = k \circ f \circ l$ is a factorization of g. Composition is defined by $(k, l)(k', l') := (k \circ k', l' \circ l)$. *Natural system of abelian groups on* $\mathscr{I}$ is a functor $\mathsf{M} : F\mathscr{I} \to \mathbf{Ab}$. All morphisms (k, l) in $F\mathscr{I}$ can be split as $(k, l) = (k, 1)(1, l)$. When considering fixed natural system M it will be of use to write $k_* := \mathsf{M}(k, 1)$ and $l^* := \mathsf{M}(1, l)$ so that, for example, $\mathsf{M}(k, l) = k_* l^*$.

The cohomology $H^*(\mathcal{I}, \mathsf{M})$ of $\mathcal{I}$ with coefficients in the natural system M is the cohomology of the cochain complex $(C^*(\mathcal{I}, \mathsf{M}), d)$, where

$$C^n(\mathsf{A}, \mathsf{M}) := \left\{ \Gamma : N_n(\mathcal{I}) \to \coprod_f \mathsf{M}f \mid \Gamma(\sigma) \in \mathsf{M}|\sigma| \right\}.$$

For $n > 1$ the coboundary d is defined by the formula ($\sigma = (f_1, \ldots, f_n)$)

$$(\mathrm{d}\Gamma)(\sigma) := (f_1)_* \Gamma(\partial_1 \sigma) + \sum_{k=2}^{n-1} (-1)^k \Gamma(\partial_k \sigma) + (-1)^n (f_n)^* \Gamma(\partial_n \sigma),$$

while for $n = 1$ and for $f : a \to b$ in $\mathcal{I}$ we let

$$(\mathrm{d}\Gamma)(f) := f_* \Gamma(a) - f^* \Gamma(b).$$

We will occasionaly write $C^*_{\mathrm{BW}}(\mathcal{I}, \mathsf{M})$ and $H^*_{\mathrm{BW}}(\mathcal{I}, \mathsf{M})$ to distinguish them from other cochain complexes and cohomology groups.

It is, of course, possible to repeat this construction for natural systems in any Abelian category, but if we choose a natural system in **Cx**, then we can enrich the complex of Baues and Wirsching with coboundary morphisms of the coefficient complexes. We show how such natural systems arise in the study of extensions of pairings.

Suppose we are given a pairing situation as in Section 3, and consider a commutative diagram in $\mathcal{B}$

$$\begin{array}{ccc} a & \xrightarrow{f} & b \\ {\scriptstyle l}\uparrow & & \downarrow{\scriptstyle k} \\ c & \xrightarrow[g]{} & d \end{array}$$

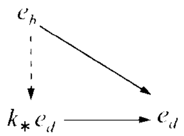
and its liftings, repectively, in $\tilde{\mathcal{E}}$ and $\mathcal{E}$:

$$\begin{array}{ccc} \tilde{e}_a & \longrightarrow & \tilde{e}_b \\ \uparrow & & \downarrow \\ \tilde{e}_c & \longrightarrow & \tilde{e}_d \end{array} \quad \text{and} \quad \begin{array}{ccc} e_a & \longrightarrow & e_b \\ \uparrow & & \downarrow \\ e_c & \longrightarrow & e_d \end{array}$$

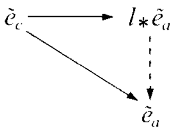
Then we have the chain of morphisms:

$$\begin{aligned} &\langle \tilde{e}_a | e_b \rangle_f = \langle \tilde{e}_a | f^* e_b \rangle_a \longrightarrow \langle l_* \tilde{e}_c | f^* e_b \rangle_a \\ &\cong \langle f_* l_* \tilde{e}_c | e_b \rangle_b \longrightarrow \langle f_* l_* \tilde{e}_c | k^* e_d \rangle_b \cong \langle k_* f_* l_* \tilde{e}_c | e_d \rangle_d \\ &\cong \langle g_* \tilde{e}_c | e_d \rangle_d = \langle \tilde{e}_c | e_d \rangle_g, \end{aligned}$$

where equalities are just definitions, isomorphisms are due to the adjointness between opcleavages and cleavages with respect to the pairing and the remaining two morphisms are obtained when we apply the bifunctoriality of the pairing to dashed morphisms in the following diagrams:



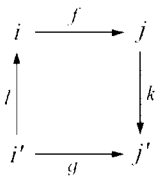
and



In this way we can construct a natural system of complexes. In fact, if we take a functor $\mathsf{B} : \mathscr{I} \to \mathscr{B}$ and its liftings $\tilde{\mathsf{E}} : \mathscr{I} \to \tilde{\mathscr{E}}$ and $\mathsf{E} : \mathscr{I} \to \mathscr{E}$, we can define the functor $\langle \tilde{\mathsf{E}} | \mathsf{E} \rangle : F\mathscr{I} \to \mathbf{Cx}$ by

$$\langle \tilde{\mathsf{E}} | \mathsf{E} \rangle (f) := \langle \tilde{\mathsf{E}} i | \mathsf{E} j \rangle_{\mathsf{B} f} \quad (f : i \to j)$$

and to every map



from f to g in $F\mathscr{I}$ we assign the cochain map between the complex $\langle \tilde{\mathsf{E}} | \mathsf{E} \rangle (f) := \langle \tilde{\mathsf{E}} i | \mathsf{E} j \rangle_{\mathsf{B} f}$ and $\langle \tilde{\mathsf{E}} | \mathsf{E} \rangle (g) := \langle \tilde{\mathsf{E}} i' | \mathsf{E} j' \rangle_{\mathsf{B} f}$ constructed as in the previous chain of morphisms. The naturality of all constructions guarantees that $\langle \tilde{\mathsf{E}} | \mathsf{E} \rangle$ is indeed a functor, namely a natural system of complexes. This fact is central to subsequent developments.

As a first application, we show how to obtain the ordinary cohomology with coefficients in a natural system using extensions of pairings. Obviously, the usual natural system of Abelian groups can be viewed as a natural system of complexes which are concentrated in grade zero. Now suppose we are given a small category $\mathscr{I}$ and a natural system of Abelian groups M. We take $\mathscr{I}$ as a base category and $(\mathscr{I}^2, \mathsf{P}_t, \mathscr{I})$ and $(\mathscr{I}^2, \mathsf{P}_s, \mathscr{I})$ as opfibration and fibration, respectively. It is easy to see that the following formulas define a pairing:

$$\langle i \xrightarrow{f} j | j \xrightarrow{g} k \rangle_j := \mathsf{M}(gf),$$

for $(h, 1) : g \to hg$ in $(\mathscr{I}^2)_j$

$$\langle f | (h, 1) \rangle_j := h_*$$

and for $(1, h) : f \to fh$ in $(\mathscr{I}^2)_j$

$$\langle (1, h) | f \rangle_j := h^*.$$

If we extend this pairing to functors from $\mathcal{I}$ and in particular for functors $\tilde{\mathsf{E}} : \mathcal{I} \to \mathcal{I}^2$ and $\mathsf{E} : \mathcal{I} \to \mathcal{I}^2$ given by $\tilde{\mathsf{E}}i := \mathsf{E}i := id_i$ and $\tilde{\mathsf{E}}f := \mathsf{E}f := ((f,f) : id_i \to id_i)$, we obtain exactly what we looked for:

Proposition 2.

$$H^*(\mathcal{I},\mathsf{M}) \cong H^*(\langle \tilde{\mathsf{E}}|\mathsf{E}\rangle_{\mathsf{Id}})$$

Proof. The proof is easy; we just show that the extension of the given pairing produces the right cochain complex. For $\sigma = (f_1,\ldots,f_n)$ we have

$$\langle \tilde{\mathsf{E}}\underline{\sigma}|\mathsf{E}\overline{\sigma}\rangle_\sigma = \mathsf{M}|\sigma|,$$

therefore,

$$C^{0,q} = \{\Gamma : N_q\mathcal{I} \to \coprod_{f:i\to j} \mathsf{M}f \mid \Gamma(\sigma) \in \mathsf{M}|\sigma|\}$$

and $C^{p,q} = 0$ for $p > 0$. It follows that

$$\langle \tilde{\mathsf{E}}|\mathsf{E}\rangle^*_{\mathsf{Id}} = (C^{0,*}, d_{\mathrm{S}}).$$

Checking the definitions of maps α_σ and β_σ (cf. Section 3) yields $\alpha_\sigma = f_{1*}$ and $\beta_\sigma = f_n^*$ hence the cochain complexes $C^*_{\mathrm{BW}}(\mathcal{I},\mathsf{M})$ and $\langle \tilde{\mathsf{E}}|\mathsf{E}\rangle^*_{\mathsf{Id}}$ are indeed equal. □

6. Natural systems of complexes

As we saw in the previous section, every pair of functors $\tilde{\mathsf{E}}$, E, with the same projection B, determines a natural system of complexes $\langle \tilde{\mathsf{E}}|\mathsf{E}\rangle$. However, the usual cochain complexes for a natural system, as defined by Baues and Wirsching are not the same as the ones defined from extension of pairings. In fact, the later exploits the 'internal' coboundaries of coefficient complexes. This motivates our next definition.

Given a small category $\mathcal{I}$ and a natural system of complexes $\mathsf{M} : F\mathcal{I} \to \mathbf{Cx}$ we define the cohomology of $\mathcal{I}$ with coefficients in M as the cohomology of the total complex of the following bicomplex:

$$C^{p,q} := \{\Gamma : N_q\mathcal{I} \to \coprod_{f:i\to j} (\mathsf{M}f)^p \mid \Gamma(\sigma) \in (\mathsf{M}|\sigma|)^p\},$$

$$(d_{\mathrm{S}}\Gamma)(\sigma) := f_{1*}\Gamma(\partial_1\sigma) + \sum_{k=2}^{n-1}(-1)^k\Gamma(\partial_k\sigma) + (-1)^n f_n^*\Gamma(\partial_n\sigma),$$

$$(d_{\mathrm{P}}\Gamma)(\sigma) := (-1)^\sigma(\Gamma\sigma),$$

where $\sigma = (f_1,\ldots,f_n)$ and δ is coboundary of the appropriate coefficient complex. Routine calculations show that it is indeed a bicomplex, so we can define

$$C^*(\mathcal{I},\mathsf{M}) := Tot(C^{*,*}, d_{\mathrm{S}}, d_{\mathrm{P}})$$

and

$$H^*(\mathscr{I},\mathsf{M}) := H(C^*(\mathscr{I},\mathsf{M}))$$

which extends the usual construction of [1].

This construction is equivalent to extension of pairings. More precisely, we have

$$C^*(\mathscr{I},\langle\hat{\mathsf{E}}|\mathsf{E}\rangle) = \langle\hat{\mathsf{E}}|\mathsf{E}\rangle_{\mathsf{B}}.$$

It clearly suffices to show that morphisms f_{1*} and f_n^* equal α_σ and β_σ, respectively. Consider for example $f_{1*} = \mathsf{M}(f,1)$. It is defined as the composition of morphisms in the following diagram:

$$\mathsf{M}u = \langle\hat{\mathsf{E}}i_q|\mathsf{E}i_1\rangle_u \cong \langle u_*\hat{\mathsf{E}}i_q|\mathsf{E}i_1\rangle \longrightarrow \langle u_*\hat{\mathsf{E}}i_q|f_1^*\mathsf{E}i_0\rangle \cong \langle\hat{\mathsf{E}}i_q|\mathsf{E}i_0\rangle_v = \mathsf{M}v,$$

where $\sigma = (i_0 \xrightarrow{f_1} i_1 \longrightarrow \cdots \xrightarrow{f_n} i_q)$, $u := \mathsf{B}|\partial_1\sigma|$, $v := \mathsf{B}|\sigma|$ so that $f_1 u = v$. This compositum is exactly the same as in the definition of α_σ. The proof that $f_n^* = \beta_\sigma$ is similar.

We use this fact to do some calculations in the general setting. Given a small category $\mathscr{I}$ and a natural system of complexes $\mathsf{M} : F\mathscr{I} \to \mathbf{Cx}$ the cohomology $H^*(\mathscr{I},\mathsf{M})$ is obtained from the double complex $C^{*,*}(\mathscr{I},\mathsf{M})$ so there is a spectral sequence converging to it. It turns out that its E_2-term can be described by means of the ordinary Baues–Wirsching cohomology.

Consider the spectral sequence associated to the double complex $(C^{*,*},d_{\mathrm{P}},d_{\mathrm{S}})$. Its E_2-term is

$$E_2^{p,q} = H_{\mathrm{S}}^q H_{\mathrm{P}}^p(C^{*,*},d_{\mathrm{P}},d_{\mathrm{S}}).$$

The coboundary d_{P} is independent of the other direction, so we can approximatively identify $H^p(C^{*,*},d_{\mathrm{P}})$ with $C^*(\mathscr{I},H^p(\mathsf{M},\delta))$ and that motivates the following construction. Every natural system of complexes $\mathsf{M} : F\mathscr{I} \to \mathbf{Cx}$ determines a natural system of graded groups $\mathsf{H} : F\mathscr{I} \to \mathbf{GrAb}$ defined as

$$\mathsf{H}^p(f) := H^p(\mathsf{M}f) \quad \text{and} \quad \mathsf{H}^p(k,l) := H^p(\mathsf{M}(k,l)).$$

Than we have

$$H^p(C^{*,*},d_{\mathrm{P}}) \cong C^*(\mathscr{I},\mathsf{H}^p).$$

Calculating the cohomology in the simplicial direction finally yields

$$E_2^{p,q} \cong H_{\mathrm{BW}}^q(\mathscr{I},\mathsf{H}^p) \Rightarrow H^*(\mathscr{I},\mathsf{M}).$$

We give two applications of this spectral sequence based on results in [1].

Given a functor $\mathsf{F} : \mathscr{I} \to \mathscr{I}'$ between small categories we can pull back every natural system of complexes $\mathsf{M} : F\mathscr{I} \to \mathbf{Cx}$ to the natural system $\mathsf{F}^*(\mathsf{M}) : F\mathscr{I}' \to \mathbf{Cx}$ defined by $\mathsf{F}^*(\mathsf{M}) := \mathsf{M}(\mathsf{F}f)$. This induces a homomorphism $\mathsf{F}^* : H^*(\mathscr{I}, \mathsf{M}) \to H^*(\mathscr{I}', \mathsf{F}^*\mathsf{M})$ which acts on cochains as $(\mathsf{F}^*\Gamma)(\sigma) := \Gamma(\mathsf{F}\sigma)$.

Proposition 3. *If* $\mathsf{F} : \mathscr{I} \to \mathscr{I}'$ *is an equivalence of categories then it induces the isomorphism*

$$\mathsf{F}^* : H^*(\mathscr{I}, \mathsf{M}) \cong H^*(\mathscr{I}', \mathsf{F}^*\mathsf{M})$$

for all natural systems of complexes M *on* $\mathscr{I}$.

Proof. The functor F induces a mapping of respective (first quadrant) spectral sequences so it would suffice to show that it induces an isomorphism between E_2-terms. We have

$$E_2^{p,q}(\mathscr{I}, \mathsf{M}) \cong H^q_{\mathrm{BW}}(\mathscr{I}, \mathsf{H}^p)$$

and

$$E_2^{p,q}(\mathscr{I}', \mathsf{F}^*\mathsf{M}) \cong H^q_{\mathrm{BW}}(\mathscr{I}', \tilde{\mathsf{H}}^p),$$

where $\tilde{\mathsf{H}}^p(f) := H^p(\mathsf{F}^*\mathsf{M}(f))$, but

$$H^p(\mathsf{F}^*\mathsf{M}(f)) = H^p(\mathsf{M}(\mathsf{F}f)) = \mathsf{H}^p(\mathsf{F}f) = \mathsf{F}^*\mathsf{H}^p(f),$$

therefore,

$$E_2^{p,q}(\mathscr{I}', \mathsf{F}^*\mathsf{M}) \cong H^q_{\mathrm{BW}}(\mathscr{I}', \mathsf{F}^*\mathsf{H}^p).$$

The induced homomorphism $H^q_{\mathrm{BW}}(\mathscr{I}, \mathsf{H}^p) \to H^q_{\mathrm{BW}}(\mathscr{I}', \mathsf{F}^*\mathsf{H}^p)$ is an isomorphism, as shown in [1, Theorem 1.11], hence their abutments are also isomorphic. □

As an immediate consequence we have (in the notation of Section 3):

Corollary 4. *Every equivalence of small categories* $\mathsf{F} : \mathscr{I} \to \mathscr{I}'$ *induces an isomorphism*

$$\langle \tilde{\mathsf{E}} | \mathsf{E} \rangle_{\mathsf{B}} \cong \langle \tilde{\mathsf{E}} \circ \mathsf{F} | \mathsf{E} \circ \mathsf{F} \rangle_{\mathsf{B} \circ \mathsf{F}}.$$

7. The cohomology of spaces

We now show how the extension of pairings, applied in a fairly simple situation, yields the cohomology groups of topological spaces.

If we consider the set of natural numbers as an ordered category then its nerve is a simplicial complex whose geometric realization is an infinite-dimensional complex Δ_∞ that can be embedded in the Hilbert cube. Denote by Δ the ordered category whose objects are the simplexes of Δ_∞ and whose morphisms are inclusion maps. The categories Δ and **Ab** are, respectively, opfibration and fibration in the trivial way over the one object – one morphism category **1**. We define the pairing between Δ and **Ab** over **1** as follows: for $\sigma \in \Delta$ and $A \in \mathbf{Ab}$

$$\langle \sigma | A \rangle := C^*_{\text{simp}}(\sigma; A),$$

where $C^*_{\text{simp}}(\sigma; A)$ is the simplicial cochain complex of σ with coefficients in A. If $i : \tau \hookrightarrow \sigma$ is a morphism in Δ then

$$\langle i | A \rangle : C^*_{\text{simp}}(\sigma; A) \to C^*_{\text{simp}}(\tau; A)$$

is just the restriction, while for $f : A \to B$ in **Ab**

$$\langle \sigma | f \rangle := C^*_{\text{simp}}(\sigma; A) \to C^*_{\text{simp}}(\sigma; B)$$

is the composition with f. For every at most countable, locally finite simplicial complex Σ its geometric realization can be simplicially embedded in Δ, hence we obtain a functor I from the ordered category Σ to Δ. A functor $\mathsf{A} : \Sigma \to Ab$ is a local system of coefficients on Σ. It is easy to check that the application of the extension of pairings yields

$$H^*(\langle \mathsf{I} | \mathsf{A} \rangle) \cong H^*(\Sigma; A),$$

the simplicial cohomology of Σ with coefficients in the local system A.

By modifying slightly the previous pairing we obtain another interesting result. For $\sigma \in \Delta$ and $A \in \mathbf{Ab}$ we define the pairing by

$$\langle \sigma | A \rangle := C^*_{\text{sing}}(\sigma; A),$$

where $C^*_{\text{sing}}(\sigma; A)$ is the singular cochain complex of σ with coefficients in A. Once again the inclusion of simplexes induces the restriction of cochains and a homomorphism f of coefficient groups induces the composition with f. If X is any topological space and $\mathscr{U}$ is a locally finite countable cover of X then its nerve $N(\mathscr{U})$ is a simplicial complex that can be simplicially embedded in Δ_∞. This yields an embedding functor $\mathsf{I} : \mathscr{I} \to \Delta$, where $\mathscr{I}$ is the ordered category of the simplicial complex $N(\mathscr{U})$. On the other side, a local system of coefficients for the covering $\mathscr{U}$ is just a functor $\mathsf{A} : \mathscr{I} \to \mathbf{Ab}$. Then $\langle \mathsf{I} | \mathsf{A} \rangle$ is the total complex of the bicomplex

$$C^{p,q} := \{\Gamma : \Sigma_q \mathscr{I} \to \coprod_{\sigma \leq \tau} C^*_{\text{sing}}(\sigma; \mathsf{A}\tau) \mid \Gamma(\sigma_0 \geq \cdots \geq \sigma_q) \in C^p(\sigma_q; \mathsf{A}\sigma_0)\},$$

with d_{P} the coboundary map of various cochain complexes and $d_{\mathrm{S}} : C^{p,q} \to C^{p,q+1}$ given by

$$(d_S\Gamma)(\sigma_0 \geq \cdots \geq \sigma_q) := \mathsf{A}_{\sigma_0\sigma_1}(\Gamma(\sigma_1 \geq \cdots \geq \sigma_{q+1})) + \sum_{i=1}^{q} (-1)^i \Gamma(\ldots \widehat{\sigma}_i \ldots)$$
$$+ (-1)^{q+1} r_{\sigma_q\sigma_{q+1}}(\Gamma(\sigma_0 \geq \cdots \geq \sigma_q)),$$

where $r_{\sigma_q\sigma_{q+1}} : C^p(\sigma_q; \mathsf{A}\sigma_0) \to C^p(\sigma_{q+1}; \mathsf{A}\sigma_0)$ is the restriction homomorphism. But this is exactly the bicomplex of the Čech cohomology with coefficients in the local system A for the cover $\mathscr{U}$, as given in [2]. If the cover is good (cf. [2]) we recover the cohomology of X with coefficients in A.

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